ASYMPTOTIC BEHAVIOUR OF ITERATES OF VOLTERRA OPERATORS ON $L^p(0,1)$

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ABSTRACT. Given $k \in L^1(0,1)$ satisfying certain smoothness and growth conditions at 0, we consider the Volterra convolution operator V_k defined on $L^p(0,1)$ by

$$(V_k u)(t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s,$$

and its iterates $(V_k^n)_{n\in\mathbb{N}}$. We construct some much simpler sequences which, as $n \to \infty$, are asymptotically equal in the operator norm to V_k^n. This leads to a simple asymptotic formula for $\|V_k^n\|$ and to a simple 'asymptotically extremal sequence'; that is, a sequence (u_n) in $L^p(0,1)$ with $\|u_n\|_p = 1$ and $\|V_k^n u_n\| \sim \|V_k^n\|$ as $n \to \infty$. As an application, we derive a limit theorem for large deviations, which appears to be beyond the established theory.

1. INTRODUCTION

A number of authors have recently published results on the asymptotic behaviour of iterated Volterra operators on $L^2(0,1)$. Lao and Whitley [4] established a number of estimates and provided numerical evidence for a conjecture about the operator norm of the Riemann-Liouville fractional integration operator which was subsequently proved by Kershaw [3] and by Little and Reade [5]. A somewhat stronger result was also independently established by Thorpe [7]. These results were generalised by the author to other Volterra convolution operators on $L^2(0,1)$ and to some extent to other Schatten-von Neumann norms in [1].

We show here that analogues of most of these L^2 results also hold in L^p. The main result, Theorem 4.3, is that if $k(t) = t^r f(t)$ where $r > -1$ and f is differentiable at 0, and we define

$$(V_k u)(t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s,$$

then the asymptotic behaviour of V_k^n is the same as that of V_h^n where

$$h(t) = f(0)t^r \mathrm{e}^{(k'(0)/k(0))t}.$$

The significance of this kernel is that there is a simple formula for its convolution powers, which leads to another asymptotically equivalent sequence of operators of rank 1 (Corollary 3.4), and an asymptoptic formula for the operator norm:

$$\|V_k^n\| \sim \frac{C_p(|f(0)|\Gamma(r+1))^n \mathrm{e}^{f'(0)/f(0)}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$, where C_p is a constant depending only on p, defined below.

As an application of these results, we derive a limit theorem for large deviations (Section 5). The exact asymptotic formula for V_k^n may also have other applications: for example, it has recently been shown [2] that the Volterra operator V with kernel 1 is not supercyclic on any L^p space; since the proof depends on direct calculations on the iterates V^n, the results and techniques established below might lead to more general results on the same lines.

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2. Notation

The term ‘sequence’ will be applied equally to sequences indexed by natural numbers or to generalised sequences indexed by positive real numbers.

Throughout, p will denote a real number in the range $[1, \infty]$ and q its Hölder conjugate, so $1/p + 1/q = 1$ if $1 < p < \infty$ and 1 is conjugate to ∞. We use $\|\cdot\|_p$ to denote the norm on L^p and the norm in the algebra of bounded operators acting on L^p. The duality pairing between $L^p(0,1)$ on $L^q(0,1)$ will be written using angle brackets: $\langle f, g\rangle = \int_0^1 fg$. We denote by C_p the constant

$$C_p = \begin{cases} \frac{1}{p^{1/p}q^{1/q}} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = 1 \text{ or } p = \infty. \end{cases}$$

Convolution of suitable functions on $[0, 1]$ is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$$

for $t \in [0, 1]$ and the n-fold convolution power of f is denoted by f^{*n}. For $k \in L^1(0, 1)$, the Volterra convolution operator V_k associated with k is defined on $L^p(0, 1)$ by $V_k f = k * f$; it is well known that for any p, V_k is a bounded operator on $L^p(0, 1)$ with operator norm $\|V_k\|_p \le \|k\|_1$.

If (a_n) and (b_n) are sequences of numbers, we shall say as usual that (a_n) and (b_n) are asympotically equal, written $a_n \sim b_n$ as $n \to \infty$, if $a_n/b_n \to 1$ as $n \to \infty$.

Extending this idea to vectors, if (u_n) and (v_n) are sequences in a normed linear space, we shall say that $u_n \sim v_n$ as $n \to \infty$ if

$$\frac{\|u_n - v_n\|}{\|u_n\|} \to 0.$$

It is easy to check that this is an equivalence relation. If (T_n) is a sequence of bounded operators on a normed linear space and (u_n) a sequence of non-zero vectors, we shall call (u_n) *asymptotically extremal* for (T_n) if $\|T_n u_n\| \sim \|T_n\|\|u_n\|$ as $n \to \infty$. We shall make frequent use of the following simple facts:

2.1. **Lemma.**

(1) *If (u_n) and (v_n) are sequences in a normed space X and $u_n \sim v_n$ as $n \to \infty$, then $\|u_n\| \sim \|v_n\|$ as $n \to \infty$;*
(2) *if in addition (S_n) and (T_n) are sequences of bounded linear operators on X such that $S_n \sim T_n$ as $n \to \infty$ and (u_n) is asymptotically extremal for (S_n) then (v_n) is asymptotically extremal for (T_n).*

For sequences of positive real numbers we also use the notation $a_n \lesssim b_n$ as $n \to \infty$ to mean that $\limsup_{n\to\infty} a_n/b_n \le 1$, and $a_n \gtrsim b_n$ as $n \to \infty$ to mean that $\liminf_{n\to\infty} a_n/b_n \ge 1$.

3. Kernels of the form $t^r \mathrm{e}^{\mu t}$

It is easy to check using the Laplace transform that if $k(t) = t^r \mathrm{e}^{\mu t}$ for some $r, \mu \in \mathbb{R}$ with $r > -1$, then the n-fold convolution power of k is given by

$$k^{*n}(t) = \frac{(\Gamma(r+1))^n}{\Gamma((r+1)n)} t^{(r+1)n-1} \mathrm{e}^{\mu t}$$

and we can choose to make this the definition of k^{*n} for non-integer $n > 0$. For such kernels we can approximate k^{*n} by operators of rank 1 and thus obtain asymptotic results. In fact, we need only consider $k_0(t) = \mathrm{e}^{\mu t}$, because

$$k_0^{*n}(t) = \frac{1}{\Gamma(n)} t^{n-1} \mathrm{e}^{\mu t}$$

so $k^{*n} = (\Gamma(r+1))^n k_0^{*(r+1)n}$.

Throughout this section, S_λ and T_λ denote the operators on $L^p(0,1)$ defined for any $\lambda \in \mathbb{R}$ by

$$(S_\lambda f)(t) = \int_0^1 \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s$$
$$(T_\lambda f)(t) = \int_0^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s.$$

We also write e_λ for the function $t \mapsto \mathrm{e}^{\lambda t}$.

3.1**. Lemma.** *For any $p \in [1,\infty]$,*

$$\|S_\lambda\|_p \sim C_p \frac{\mathrm{e}^\lambda}{\lambda}$$

as $\lambda \to \infty$ through $\mathbb{R}^+$. (The constant C_p is defined in Section 2.) If f_λ is defined by

$$f_\lambda(t) = \begin{cases} \mathrm{e}^{-g(\lambda)\lambda t} & \text{if } p = 1 \\ \mathrm{e}^{-\lambda t/(p-1)} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \end{cases}$$

where g is any function such that $g(\lambda) \to \infty$ as $\lambda \to \infty$, then (f_λ) is asymptotically extremal for (S_λ).

Proof. We can write S_λ in the form

$$(S_\lambda f)(t) = \mathrm{e}^{\lambda t} \int_0^1 \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s = \langle f, e_{-\lambda}\rangle\, e_\lambda$$

from which we see immediately that $\|S_\lambda\| = \|e_\lambda\|_p \|e_{-\lambda}\|_q$ and an easy calculation leads to the asymptotic formula given above.

If $p > 1$ then f_λ is taken directly from the extremal case of Hölder's inequality. If $p = 1$ then there is no exact extremal function for S_λ, but it is a simple calculation to check that f_1 is asymptotically extremal for any g tending to ∞ at ∞. □

3.2**. Lemma.** *For any $p \in [1,\infty]$, the sequences of operators (S_λ) and (T_λ) defined above are asymptotically equal as $\lambda \to \infty$ through $\mathbb{R}^+$. In particular, $\|T_\lambda\|_p \sim C_p \mathrm{e}^\lambda/\lambda$.*

Proof. Intuitively, S_λ and T_λ are close to each other for large λ because their kernels differ only in the region $s > t$, where $\mathrm{e}^{\lambda(t-s)}$ is small when λ is large. We can estimate the rate of decay of $\|S_\lambda - T_\lambda\|$ as follows:

$$\begin{aligned} ((S_\lambda - T_\lambda)f)(t) &= \int_t^1 \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s \\ &= \int_0^{1-t} \mathrm{e}^{\lambda(t-1+u)} f(1-u)\,\mathrm{d}u \\ &= \int_0^{1-t} \mathrm{e}^{-\lambda(1-t-u)} f(1-u)\,\mathrm{d}u \\ &= (e_{-\lambda} * Rf)(1-t) \end{aligned}$$

where R is the operator on $L^p(0,1)$ defined by $(Rf)(t) = f(1-t)$. We thus have that

$$S_\lambda - T_\lambda = RV_{e_{-\lambda}}R.$$

Now, R is an isometric bijection on $L^p(0,1)$, so $\|S_n - T_n\|_p = \|V_{e_{-\lambda}}\|_p$. We can now use the standard estimate to see that

$$\|V_{e_{-\lambda}}\|_p \le \int_0^1 \mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{1-\mathrm{e}^{-\lambda}}{\lambda} \sim \frac{1}{\lambda}$$

so by Lemma 3.1, $\|S_\lambda - T_\lambda\|/\|S_\lambda\| \lesssim C_p^{-1}\mathrm{e}^{-\lambda}$ as $\lambda \to \infty$. This shows that $S_\lambda \sim T_\lambda$ as $\lambda \to \infty$, so $\|T_\lambda\|_p \sim \|S_\lambda\|_p \sim C_p \mathrm{e}^\lambda/\lambda$ by Lemma 3.1. □

3.3**. Lemma.** *For some fixed $\mu \in \mathbb{R}$ and $p \in [1, \infty]$, let $k(t) = \mathrm{e}^{\mu t}$ and consider the Volterra operator V_k acting on $L^p(0,1)$. Then*

$$V_k^n \sim \frac{\mathrm{e}^{-(n-1)}}{\Gamma(n)} S_{n-1+\mu}$$

and in particular

$$\|V_k^n\|_p \sim \frac{C_p \mathrm{e}^{\mu}}{\Gamma(n+1)}$$

as $n \to \infty$ through $\mathbb{R}^+$.

Proof. We have

$$\begin{aligned}\left\| \Gamma(n) V_k^n - \mathrm{e}^{-(n-1)} T_{n-1+\mu} \right\|_p &\leq \int_0^1 |t^{n-1}\mathrm{e}^{\mu t} - \mathrm{e}^{-(n-1)}\mathrm{e}^{(n-1+\mu)t}| \,\mathrm{d}t \\ &= \int_0^1 \mathrm{e}^{-(n-1)}\mathrm{e}^{(n-1+\mu)t} - t^{n-1}\mathrm{e}^{\mu t} \,\mathrm{d}t\end{aligned}$$

(since $t^{n-1}\mathrm{e}^{\mu t} \leq \mathrm{e}^{-(n-1)}\mathrm{e}^{(n-1+\mu)t}$ for $t \in [0,1]$)

$$\begin{aligned}&\leq \mathrm{e}^{\mu} \int_0^1 \mathrm{e}^{(n-1)(t-1)} - t^{n-1} \,\mathrm{d}t \\ &< \mathrm{e}^{\mu} \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{\mathrm{e}^{\mu}}{n(n-1)}\end{aligned}$$

$$\frac{\|\Gamma(n)V_k^n - \mathrm{e}^{-(n-1)}T_{n-1+\mu}\|_p}{\|\mathrm{e}^{-(n-1)}T_{n-1+\mu}\|_p} \lesssim \frac{\mathrm{e}^{\mu}/n(n-1)}{\mathrm{e}^{-(n-1)}C_p\mathrm{e}^{n-1+\mu}/n} = \frac{1}{C_p(n-1)} \to 0.$$

using Lemma 3.2. This shows that $\Gamma(n)V_k^n \sim \mathrm{e}^{-(n-1)}T_{n-1+\mu}$ as $n \to \infty$. But $S_\lambda \sim T_\lambda$ as $\lambda \to \infty$ by Lemma 3.2, so

$$V_k^n \sim \frac{\mathrm{e}^{-(n-1)}}{\Gamma(n)} S_{n-1+\mu}$$

as claimed. The asymptotic formula for $\|V_k^n\|$ now follows from Lemma 2.1 and Lemma 3.1. □

3.4**. Corollary.** *Fix $\mu, r \in \mathbb{R}$ with $r > -1$, let $k(t) = t^r \mathrm{e}^{\mu t}$ and consider the Volterra operator V_k acting on $L^p(0,1)$ where $1 \leq p \leq \infty$. Then*

$$V_k^n \sim \frac{\Gamma(r+1)^n \mathrm{e}^{-((r+1)n-1)}}{\Gamma((r+1)n)} S_{(r+1)n-1+\mu}$$

and in particular

$$\|V_k^n\|_p \sim \frac{C_p \mathrm{e}^{\mu}(\Gamma(r+1))^n}{\Gamma((r+1)n+1)}$$

as $n \to \infty$ through $\mathbb{R}^+$.

Proof. As remarked at the beginning of the section, if we define $k_0(t) = \mathrm{e}^{\mu t}$ then we have $k^{*n} = (\Gamma(r+1))^n k_0^{*(r+1)n}$. The result is now immediate from Lemma 3.3. □

4. MORE GENERAL KERNELS

It is easy to see that if $h, k \in L^1(0,1)$ with $0 \leq h \leq k$ then $\|V_h\|_p \leq \|V_k\|_p$ for any $p \in [1, \infty]$. This simple fact, in combination with the results from the previous section, allows us to deduce asymptotic results for a large class of kernels.

4.1. **Lemma.** *Suppose k is a measurable function on $[0,1]$ and there exist real constants c,μ,ν,r with $c>0$ and $r>1$ such that*

$$ct^r\mathrm{e}^{\mu t} \le k(t) \le ct^r\mathrm{e}^{\nu t}$$

for $t\in[0,1]$. Then for any $\delta\in(0,1)$, any $j\in\mathbb{N}$ and any polynomial P,

$$\frac{P(n)\int_0^{1-\delta} k^{*(n-j)}}{\|V_k^n\|_p} \to 0$$

as $n\to\infty$.

Proof. Taking the $(n-j)$-fold convolution power of the right-hand inequality gives

$$k^{*n-j}(t) \le \frac{(c\Gamma(r+1))^n}{\Gamma((r+1)(n-j))} t^{(r+1)(n-j)-1}\mathrm{e}^{\nu t}$$

Using the estimate $\mathrm{e}^{\nu t}\le \max(1,\mathrm{e}^{\nu})$, we have if $(r+1)(n-j)>1$,

$$\begin{aligned}\int_0^{1-\delta} k^{*(n-j)} &= \frac{(c\Gamma(r+1))^{n-j}}{\Gamma((r+1)(n-j))}\int_0^{1-\delta} t^{(r+1)(n-j)-1}\mathrm{e}^{\nu t}\,\mathrm{d}t \\ &\le \frac{(c\Gamma(r+1))^{n-j}}{\Gamma((r+1)(n-j))}\frac{(1-\delta)^{(r+1)(n-j)}\max(1,\mathrm{e}^{\nu})}{(r+1)(n-j)}.\end{aligned}$$

We can also see from the n-fold convolution power of the left-hand inequality and Corollary 3.4 that

$$\|V_k^n\|_p \gtrsim \frac{C_p\mathrm{e}^{\mu}(c\Gamma(r+1))^n}{\Gamma((r+1)n+1)}.$$

Combining these gives

$$\frac{P(n)\int_0^{1-\delta} k^{*(n-j)}}{\|V_k^n\|_p} \lesssim \frac{P(n)\max(1,\mathrm{e}^{\nu})\Gamma((r+1)n+1)(1-\delta)^{(r+1)(n-j)}}{C_p\mathrm{e}^{\mu}(c\Gamma(r+1))^j\Gamma((r+1)(n-j))(r+1)(n-j)}.$$

It is an immediate consequence of Stirling's formula that $\Gamma(n+s)/\Gamma(n)\sim n^s$ as $n\to\infty$ for any s, from which it follows that the right-hand side tends to zero as $n\to\infty$. □

We can now establish a localisation result: for a wide range of kernels, the asymptotic behaviour of V_k^n is determined by the values of k in any neighbourhood of 0.

4.2. **Lemma.** *Suppose $h,k\in L^1(0,1)$, that h and k are equal on the interval $[0,\delta]$ for some $\delta\in(0,1)$ and that there exist real constants c,μ,ν,r with $c>0$ and $r>1$ such that*

$$ct^r\mathrm{e}^{\mu t} \le h(t) \le ct^r\mathrm{e}^{\nu t}$$

for $t\in[0,1]$. Then for any $p\in[1,\infty]$, $V_k^n\sim V_h^n$ on $L^p(0,1)$ as $n\to\infty$.

Proof. Let $g=k-h$, so $k=h+g$ and g is zero on $[0,\delta]$. We can use the binomial theorem in the convolution algebra $L^1(0,1)$ to write

$$k^{*n} = (h+g)^{*n} = h^{*n} + g^{*n} + \sum_{j=1}^{n-1}\binom{n}{j} g^{*j} * h^{*(n-j)}.$$

Now, if we were working on the whole of $\mathbb{R}$, then g^{*n} would be supported on $[n\delta,n]$ and $g^{*j}*h^{*(n-j)}$ would be supported on $[j\delta,n]$. But we are working in $L^1(0,1)$, so if we choose $N>1/\delta$ then for $n\ge N$ we have

$$k^{*n} = h^{*n} + \sum_{j=1}^{N-1}\binom{n}{j} g^{*j} * h^{*(n-j)}.$$

Moreover, since g^{*j} is supported to the right of $j\delta$, we have

$$g^{*j} * h^{*(n-j)} = g^{*j} * (h^{*(n-j)}\chi_{[0,1-j\delta]})$$

and hence

$$k^{*n} - h^{*n} = \sum_{j=1}^{N-1}\binom{n}{j} g^{*j} * (h^{*(n-j)}\chi_{[0,1-j\delta]}).$$

We can therefore estimate

$$\frac{\|V_k^n - V_h^n\|_p}{\|V_h^n\|_p} \leq \sum_{j=1}^{N} \binom{n}{j} \frac{\left(\int_0^1 |g^{*j}|\right)\left(\int_0^{1-j\delta} h^{*(n-j)}\right)}{\|V_h^n\|_p}.$$

This is a finite sum of terms, all of which tend to zero by Lemma 4.1, so we can conclude that $V_h^n \sim V_k^n$ as $n \to \infty$. □

We are now in a position to prove the main result.

4.3**. Theorem.** *Suppose $k \in L^1(0,1)$ is such that $k(t) = t^r f(t)$ where $r > -1$, $f(0) \neq 0$ and $f'(0)$ exists, and let*

$$h(t) = f(0)t^r \mathrm{e}^{(f'(0)/f(0))t}.$$

Then for any $p \in [1,\infty]$, $V_k^n \sim V_h^n$ on $L^p(0,1)$. It follows that V_k^n is also asymptotically equivalent to the sequence of rank 1 operators described in Corollary 3.4; in particular:

$$\|V_k^n\|_p \sim \frac{C_p(|f(0)|\Gamma(r+1))^n \mathrm{e}^{f'(0)/f(0)}}{\Gamma((r+1)n+1)}$$

and if

$$f_n(t) = \begin{cases} \mathrm{e}^{-g(n)nt} & \textit{if } p = 1 \\ \mathrm{e}^{-((r+1)n-1+k'(0)/k(0))t/(p-1)} & \textit{if } 1 < p < \infty \\ 1 & \textit{if } p = \infty \end{cases}$$

where g is any function such that $g(n) \to \infty$ as $n \to \infty$, then (f_n) is asymptotically extremal for (V_k^n).

Proof. For $\eta \in \mathbb{R}$, let

$$h_\eta(t) = f(0)t^r \mathrm{e}^{(f'(0)/f(0)+\eta)t}.$$

We can assume without loss of generality that $f(0) > 0$, so $\log f$ is differentiable at 0 and hence for any $\eta > 0$ there exists $\delta_\eta \in (0,1)$ such that if $0 < t \leq \delta_\eta$ then

$$(\log f)'(0) - \eta \leq \frac{\log f(t) - \log f(0)}{t} \leq (\log f)'(0) + \eta$$

or equivalently

$$f(0)t^r \mathrm{e}^{(f'(0)/f(0)-\eta)t} \leq k(t) \leq f(0)t^r \mathrm{e}^{(f'(0)/f(0)+\eta)t}.$$

Now let

$$k_\eta(t) = \begin{cases} k(t) & \text{if } 0 \leq t \leq \delta_\eta \\ h(t) & \text{if } \delta_\eta < t \leq 1 \end{cases}$$

so $h_{-\eta} \leq k_\eta \leq h_\eta$ and $h_{-\eta} \leq h \leq h_\eta$. Because all the functions involved are non-negative, we can take the n-fold convolution power of these inequalities to give $h_{-\eta}^{*n} \leq k_\eta^{*n} \leq h_\eta^{*n}$ and $h_{-\eta}^{*n} \leq h^{*n} \leq h_\eta^{*n}$. If follows that $|k_\eta^{*n} - h^{*n}| \leq h_\eta^{*n} - h_{-\eta}^{*n}$ and we can integrate to give, abbreviating $f'(0)/f(0)$ to μ,

$$\begin{aligned} \|V_{k_\eta}^n - V_h^n\|_p &\leq \frac{(f(0)\Gamma(r+1))^n}{\Gamma((r+1)n)} \int_0^1 t^{(r+1)n-1}\left(\mathrm{e}^{(\mu+\eta)t} - \mathrm{e}^{(\mu-\eta)t}\right) \mathrm{d}t \\ &\leq \frac{(f(0)\Gamma(r+1))^n}{\Gamma((r+1)n)}\left(\mathrm{e}^{\mu+\eta} - \mathrm{e}^{\mu-\eta}\right) \int_0^1 t^{(r+1)n-1}\,\mathrm{d}t \\ &= \frac{(f(0)\Gamma(r+1))^n}{\Gamma((r+1)n)} \frac{1}{(r+1)n}\left(\mathrm{e}^{\mu+\eta} - \mathrm{e}^{\mu-\eta}\right) \\ &\leq \frac{K_1(f(0)\Gamma(r+1))^n \eta}{\Gamma((r+1)n+1} \end{aligned}$$

for all $n \in \mathbb{N}$ and all $\eta \in [0,1]$, say, where K_1 is a constant independent of n and η. We also have

$$k_\eta^{*n} \geq h_{-\eta}^{*n} \geq \mathrm{e}^{-\eta} h^{*n}$$

so $\|V_{k_\eta}^n\|_p \geq \mathrm{e}^{-\eta}\|V_h^n\|_p$. But

$$\|V_h^n\|_p \sim \frac{C_p(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu}}{\Gamma((r+1)n+1)}$$

so in particular

$$\|V_h^n\|_p \geq \frac{K_2(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu}}{\Gamma((r+1)n+1)}$$

for all $n \in \mathbb{N}$, where K_2 is independent of n. Combining all these, we see that

$$\frac{\|V_{k_\eta}^n - V_h^n\|_p}{\|V_{k_\eta}^n\|_p} \leq K_3 \eta \mathrm{e}^{\eta}$$

for all $n \in \mathbb{N}$ and all $\eta \in [0,1]$, where K_3 is independent of n and η.

Now, for any $\varepsilon > 0$ we can find $\eta \in (0,1)$ such that

$$\frac{\|V_{k_\eta}^n - V_h^n\|}{\|V_{k_\eta}^n\|} < \frac{\varepsilon}{2\mathrm{e}}$$

for all $n \in \mathbb{N}$. We can also use Lemma 4.2 to find $N \in \mathbb{N}$ such that if $n > N$ then

$$\frac{\|V_k^n - V_{k_\eta}^n\|_p}{\|V_{k_\eta}^n\|_p} < \frac{\varepsilon}{2\mathrm{e}}$$

and hence

$$\frac{\|V_k^n - V_h^n\|_p}{\|V_{k_\eta}^n\|_p} < \frac{\varepsilon}{\mathrm{e}}.$$

But $k_\eta^{*n} \leq h_\eta^{*n} \leq \mathrm{e}^{\eta} h^{*n} \leq \mathrm{e} h^{*n}$ since $\eta \in (0,1)$. We therefore have $\|V_{k_\eta}^n\| \leq \mathrm{e}\|V_h^n\|$, so for $n > N$ we have

$$\frac{\|V_k^n - V_h^n\|_p}{\|V_h^n\|_p} < \varepsilon$$

showing that (V_k^n) and (V_h^n) are asymptotically equal. Their norms are thus also asymptotically equal so we have

$$\|V_k^n\|_p \sim \frac{C_p(f(0)\Gamma(r+1))^n \mathrm{e}^{\mu}}{\Gamma((r+1)n+1)}$$

by Corollary 3.4, as claimed. We also know from Corollary 3.4 that

$$V_h^n \sim \frac{\Gamma(r+1)^n \mathrm{e}^{-((r+1)n-1)}}{\Gamma((r+1)n)} S_{(r+1)n-1+\mu}$$

as $n \to \infty$, where S_λ and T_λ is as defined in Section 3, so we have

$$V_k^n \sim \frac{\Gamma(r+1)^n \mathrm{e}^{-((r+1)n-1)}}{\Gamma((r+1)n)} S_{(r+1)n-1+\mu}.$$

By Lemma 2.1, these two sequences of operators have the same asymptotically extremal sequences of vectors. An appropriate sequence for (S_λ) was identified in Lemma 3.1; substituting $\lambda = (r+1)n - 1 + k'(0)/k(0)$ gives the sequence in the statement of the theorem. $\square$

5. Further remarks on the case $p = 1$: a probabilistic interpretation

In the case $p = 1$, the estimate used throughout is in fact exact: $\|V_k\|_1 = \|k\|_1$ (consider the action of V_k on an approximate identity). Theorem 4.3 thus gives the following result about powers of elements of the Volterra algebra $L^1(0,1)$:

5.1. **Corollary.** *Suppose $k \in L^1(0,1)$ is such that $k(t) = t^r f(t)$ where $r > -1$, $f(0) \neq 0$ and $f'(0)$ exists. Then*

$$\|k^{*n}\|_1 \sim \frac{(|f(0)|\Gamma(r+1))^n \mathrm{e}^{f'(0)/f(0)}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$.

If $k \in L^1(0,\infty)$, $k \geq 0$ a.e. and $\int_0^\infty k = 1$ then we can interpret k as the probability density of a random variable and k^{*n} as the density of the sum of n independent random variables with density k. The L^1 norm of the restriction to $(0,1)$ of k^{*n} is then the probability that this sum is no larger than 1.

5.2**. Corollary.** *Suppose $k \in L^1(0,\infty)$ is a probability density function and that $k(t) = t^r f(t)$ where $r > -1$, $f(0) \neq 0$ and $f'(0)$ exists. Let (X_n) be a sequence of independent random variables with this density, and let $S_n = X_1 + X_2 + \cdots + X_n$. Then*

$$\mathbf{P}(S_n \leq 1) \sim \frac{(f(0)\Gamma(r+1))^n \mathrm{e}^{f'(0)/f(0)}}{\Gamma((r+1)n+1)}$$

as $n \to \infty$.

This limit theorem seems to go beyond the scope of known results on such sums, such as those in Petrov [6, Section 5.8]. In the notation of that section, we have $x = O(n^{1/2})$ but not $x = o(n^{1/2})$ which, as explicitly noted, is not sufficient for the results there to apply.

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